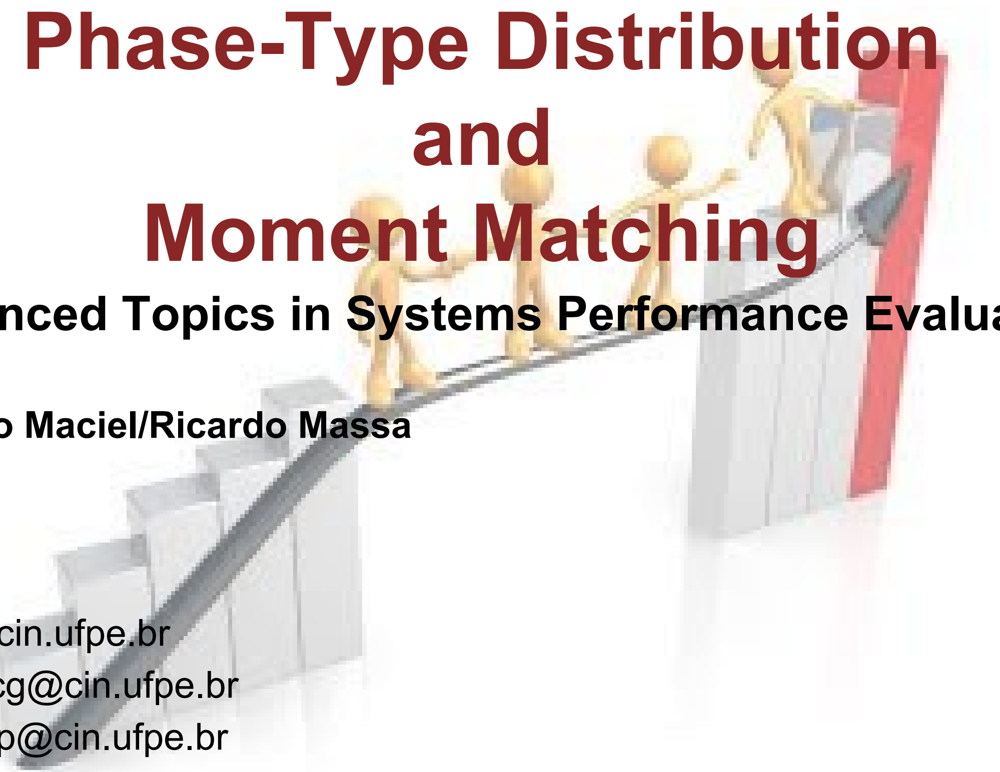
Phase-Type Distribution and Moment Matching



Advanced Topics in Systems Performance Evaluation

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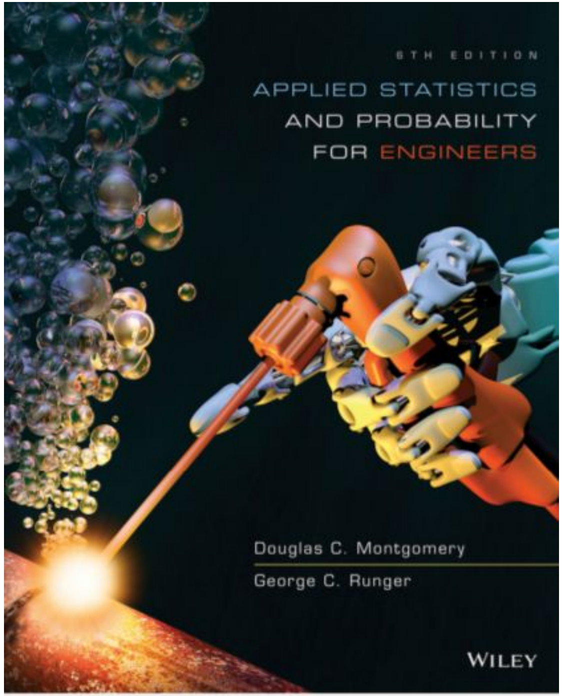
Summary

- Preliminaries
- Phase-type Definition
- Moment Matching Methods

"Statistics is a science that helps us make decisions and draw conclusions in the presence of variability".

Douglas Montgomery



Preliminaries

“In practice, totally deterministic systems are unlikely due to influence of unpredictable factors.”

Theme:

Phase-type
Distribution
And
Moment Matching

Theme:

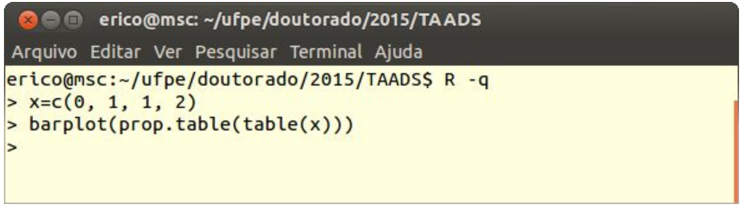
Phase-type
Distribution
And
Moment Matching


Recall...

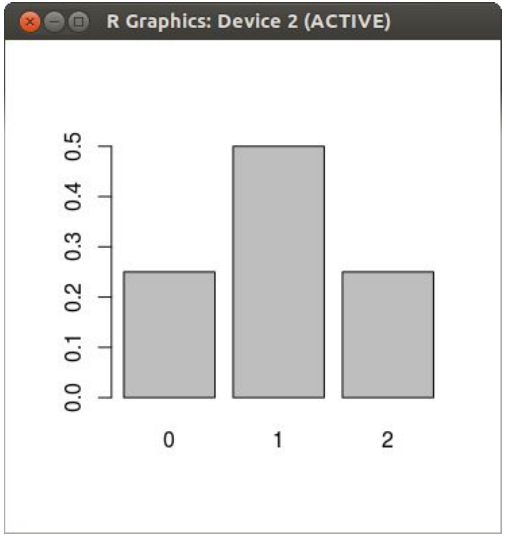
Preliminaries

a scientific **test** in which you perform a series of actions and carefully observe their effects in order to learn about something [Merriam, 2015].

- **Experiment 1: toss a coin twice**
  - **Which are the possible outcomes?**
  - **HH, HT, HT, TT**

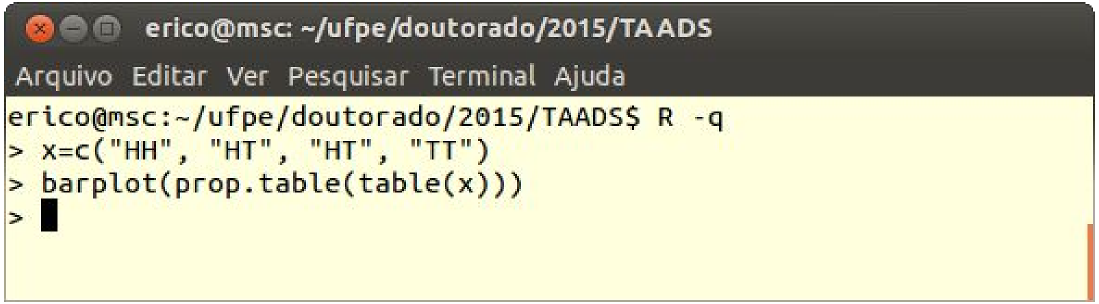


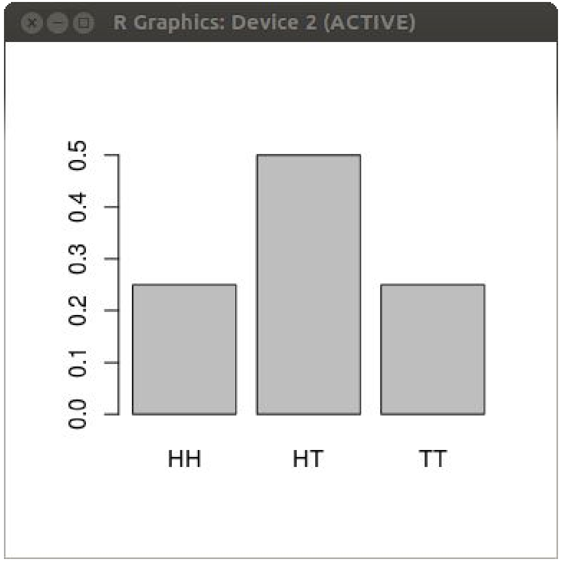

```
P(X=HH) = 0,25
P(X=HT) =  0,5
P(X=TT) = 0,25
```



Preliminaries

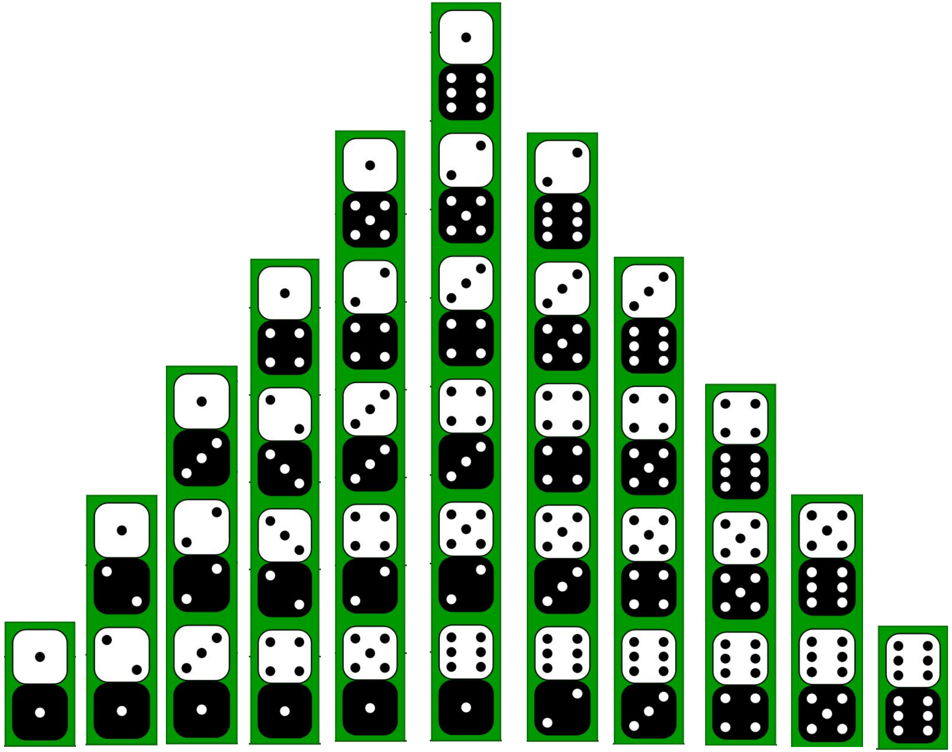
- **Experiment 1: toss a coin twice**
  - With Categorical Variable: HH, HT, HT, TT





Preliminaries

- **Experiment 2: toss two dices**



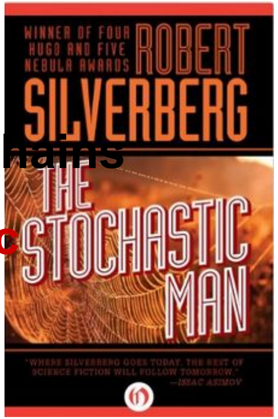
$\mathcal{S} = \{\ 2\quad 3\quad 4\quad 5\quad 6\quad 7\quad 8\quad 9\quad 10\quad 11\quad 12\ \}$

Preliminaries

- **Exponential Distributions**

It possesses the so-called memoryless property

it often leads to the introduction of Markov chains in the study of stochastic systems.



"The concepts of cause and effect are fallacies. There are only seeming causes leading to apparent effects."

"A system in which events occur according to a law of probability but aren't individually determined in accordance with the principle of causality is a stochastic system."

Preliminaries

- **Stochastic:** Merriam-Webster

Preliminaries

- **Em bom Português...**

Preliminaries

- **Exponential Distributions**
  - The sole parameter λ of the exponential distribution can be interpreted explicitly under different circumstances.
  - For instance:
    - λ is interpreted as the failure rate in reliability theory while it represents the arrival rate of the Poisson process

Preliminaries

- **Phase-type distributions were introduced by Neuts, in 1975, as a generalization of the exponential distribution.**

Since then, matrix-analytic methods have become an **indispensable tool** in stochastic modeling and have found applications in the analysis and design of :

- *manufacturing systems*
- *telecommunications networks*
- *risk/insurance models*
- *reliability models*
- *inventory and supply chain systems*

pioneered in matrix-analytic methods in the study of queueing models

Preliminaries

- **PH-type Distributions → Motivations**
  - can be defined on Markov chains
  - have a **partial memoryless property**
    - which often leads to Markovian models that are **analytically and algorithmically tractable**

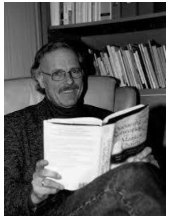
Preliminaries

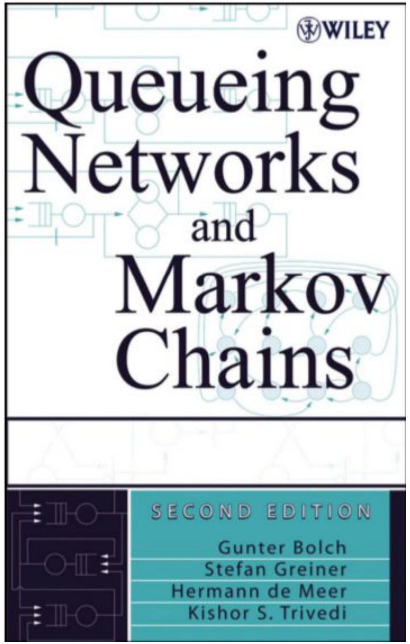
- **PH-type Distributions**

They constitute a versatile class of distributions that **can approximate arbitrarily closely any probability distribution** defined on the nonnegative real line

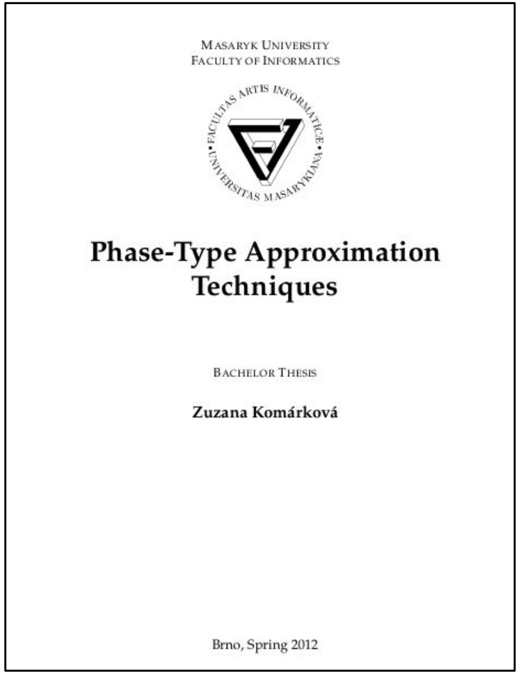
Preliminaries

- **Pointed references**





Approximation problem:
- approximating general distributions by phase-type distributions


MASARYK UNIVERSITY
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Phase-Type Approximation Techniques

BACHELOR THESIS

Zuzana Komárková

Brno, Spring 2012

Preliminaries

- **Keywords**

Semi-Markov Process(SMP)

Continuous Time Markov Chain(CTMC)

Complex Systems

Poisson Process

Generalized Semi-Markov Process(GSMP)

Stochastic Models

Preliminaries

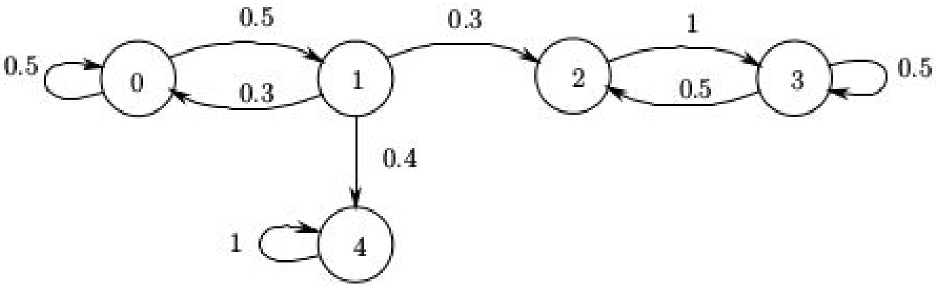
- **Cloud Terms Generators**

Preliminaries

- **Continuous-Time Markov Chain(CTMC) Definition**
    - A CTMC is a tuple($\boldsymbol{S}, \boldsymbol{R}, \boldsymbol{v_0}$), where:
        - ***S*** is a finite set of states
        - ***R***: S x S → $\mathbb{R}$ is a rate matrix, such that R(i,j) ≥ 0 for i ≠ j and R(i,j) = -$\sum_{k \neq i}$R(i,k), and
        - $\boldsymbol{v_0}$: is an initial probability vector

Preliminaries

- **CTMC - Absorbing state**
  - A state is called absorbing if once it has been reached it is impossible to leave it
  - There is any absorbing state in below CTMC?



Preliminaries

- **Named from?**
  - **The name of phase-type distributions comes from the key idea behind it, which is to model waiting times as being made up of the time taken to move through a number exponentially distributed phases.**
  - **These phases are equivalent to states of the underlying Markov chain**
    - **The term phases are used when writing about PH-distributions, and states when writing about CTMC**

Preliminaries

- **Quantitative analysis of man-made systems like:**
  - ***computer systems***
  - ***communication networks***
  - **manufacturing plants**
  - **logistics networks**
  - **(...)**

is often done by means of discrete event <u>models</u> that are analyzed numerically or by simulation

Preliminaries

- **PHD - PHase-type Distribution**
  - The approximation of a general distribution by a PHD is a complex non-linear optimization problem for which only recently (2014) computational algorithms have been proposed which have not found their way into broadly available modeling software yet.

Definition - PH Distribution

- **PHD - PHase-type Distribution**
  - Consider an absorbing CTMC with n+1 states, where n ≥ 1, such that states 1, ..., n are transient and state 0 is an absorbing state.
  - Further, let the chain have an initial probability of starting in any of the **n+1** phases given by the probability vector (p_0 **p**).
  - The (continuous) n-phase **phase-type distribution** (PH-distribution) is the **distribution of time** from the above process's starting until absorption in the absorbing state.

Definition - PH Distribution

- **Continuous-Time Markov Chain(CTMC):**
    - Infinitesimal Generator Matrix

$$\mathbf{Q} = \begin{array}{cc} \begin{array}{cc} \overbrace{\hphantom{\mathbf{D}_0}}^{n} & \overbrace{\hphantom{\mathbf{d}_1}}^{1} \end{array} & \\ \left[\begin{array}{c|c} \mathbf{D}_0 & \mathbf{d}_1 \\ \hline \mathbf{0} & 0 \end{array}\right] & \begin{array}{l} \}\,n \\ \}\,1 \end{array} \end{array}$$

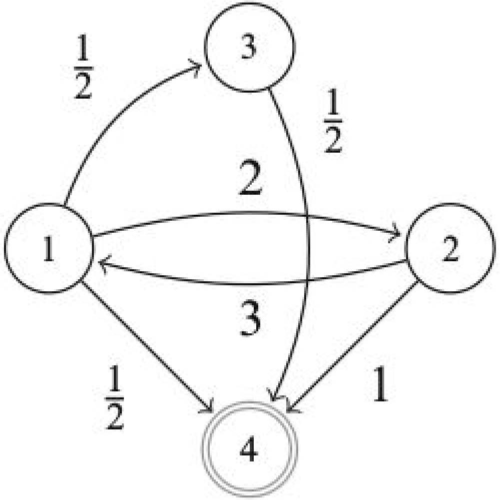
$\mathbf{D}_0$: submatrix describing only transitions between transient states

$\mathbf{d}_1$: transitions from transient states to the absorbing state

$\mathbf{0}$: contains 0's(no transitions from the absorbing states to transitions states can occur)

Definition - PH Distribution

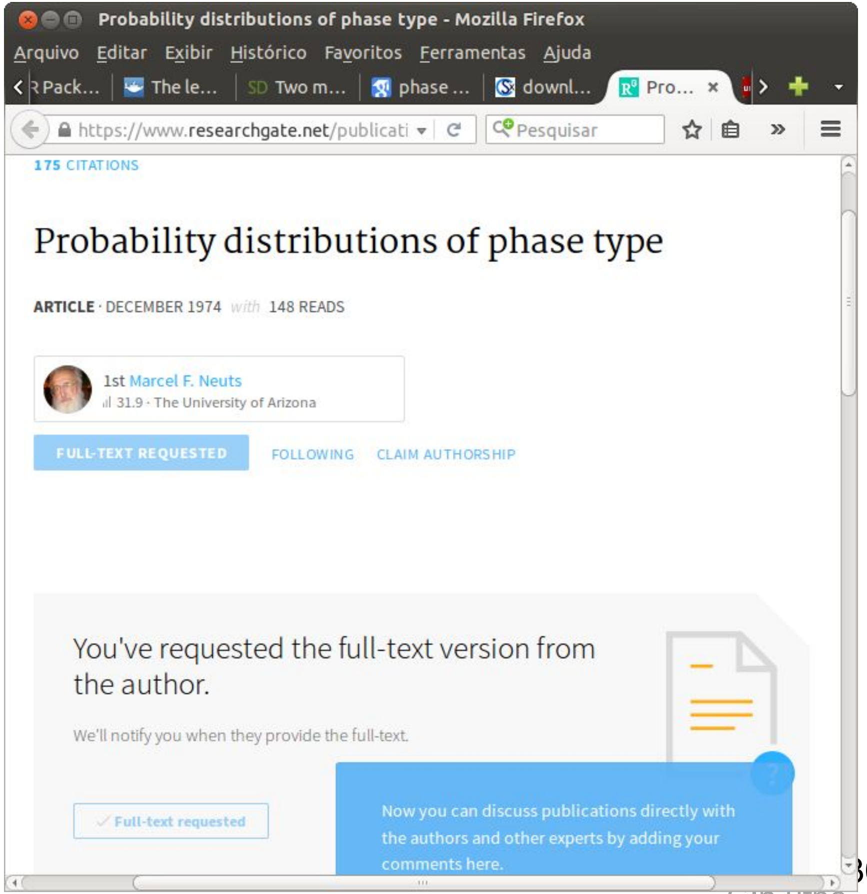
- **Continuous-Time Markov Chain(CTMC):**
  - Example: What is the Infinitesimal Generator Matrix $\mathbf{Q}$ of below CTMC?



$$\mathbf{Q} = \left[\begin{array}{ccc|c} -3 & 2 & \frac{1}{2} & \frac{1}{2} \\ 3 & -4 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

Definition - PH Distribution

- **Neuts' definition**



Definition - PH Distribution

- **Neuts' definition**

1981 Winter Simulation Conference Proceedings
T.I. Ören, C.M. Delfosse, C.M. Shub (Eds.)

GENERATING RANDOM VARIATES FROM A DISTRIBUTION OF PHASE TYPE

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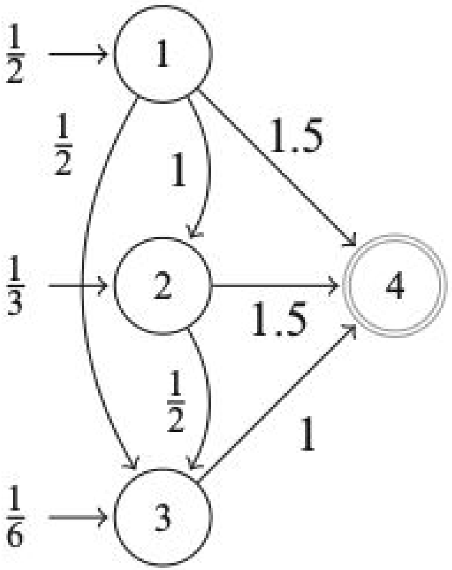
A distribution $F(\cdot)$ on $[0,\infty)$ is a PH-distribution if it is that of the time until absorption in a finite state Markov process with generator

1) $$Q = \begin{vmatrix} T & \underline{T}^{\circ} \\ \underline{0} & 0 \end{vmatrix}$$

and initial probability vector $(\underline{\alpha}, \alpha_{m+1})$. T is a non-singular matrix of order m and satisfies $T_{ii} < 0$, for $1 \leq i \leq m$, and $T_{ij} \geq 0$, for $i \neq j$. Also $T\underline{e} + \underline{T}^{\circ} = \underline{0}$ and $\underline{\alpha}\,\underline{e} + \alpha_{m+1} = 1$, where $\underline{e}$ denotes a column vector of appropriate dimension with all components equal to one. The distribution $F(\cdot)$ is then given by

Definition - PH Distribution

- **PHD - PHase-type Distribution**



$$\mathbf{D}_0 = \begin{bmatrix} -3 & 1 & 0.5 \\ 0 & -2 & 0.5 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{d}_1 = \begin{bmatrix} 1.5 \\ 1.5 \\ 1 \end{bmatrix},$$

$$\pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

Acyclic Phase-Type Distributions

- **Can be divided into various subclasses:**

Hypo-Exponential Distribution

Exponential Distribution

$$c_X = \frac{\sqrt{E[(X - m_1)^2]}}{m_1}$$

Coxian Distribution

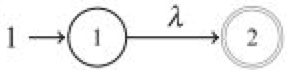
Erlang Distribution

Hyper-Exponential Distribution

Acyclic Phase-Type Distributions

- **Exponential Distribution**
  - **Simplest case of a PHD with a single transient state.**
  - **Memoryless:** $Prob(X > t + s | X > t) = Prob(X > s)$ for all $t, s \geq 0.$
  - **Mean:** $E[X] = \frac{1}{\lambda}$ **Variance:** $VAR[X] = \frac{1}{\lambda^2}$

a

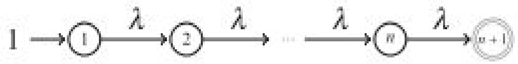


b $\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$

Acyclic Phase-Type Distributions

- **Erlang Distribution**
  - **It is a sequence of *n* exponential distributions with the same rate.**
  - **Denoted by:** $E(n, \lambda)$
  - **Initial probability vector:** $\pi = [1, 0, \ldots, 0]$

a



b

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & \lambda & \ldots & 0 & 0 \\ 0 & -\lambda & \ldots & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & 0 & \ldots & -\lambda & \lambda \\ 0 & 0 & \ldots & 0 & -\lambda \end{bmatrix}$$

Acyclic Phase-Type Distributions

- **Hypo-Exponential Distribution**
  - **Generalized Erlang Distribution.**
  - **Rates** $\lambda(1),\ldots,\lambda(n)$ **are not necessarily identical.**

a

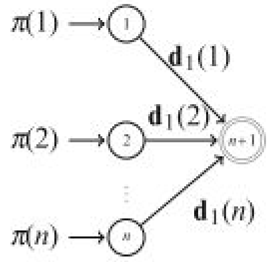
b

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda(1) & \lambda(1) & \ldots & 0 & 0 \\ 0 & -\lambda(2) & \ldots & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & 0 & \ldots & -\lambda(n-1) & \lambda(n-1) \\ 0 & 0 & \ldots & 0 & -\lambda(n) \end{bmatrix}$$

Acyclic Phase-Type Distributions

- **Hyper-Exponential Distribution**
  - **It is a convex mixture of *n* exponential distributions.**
  - $\pi(i) > 0$ **for all phases.**

a

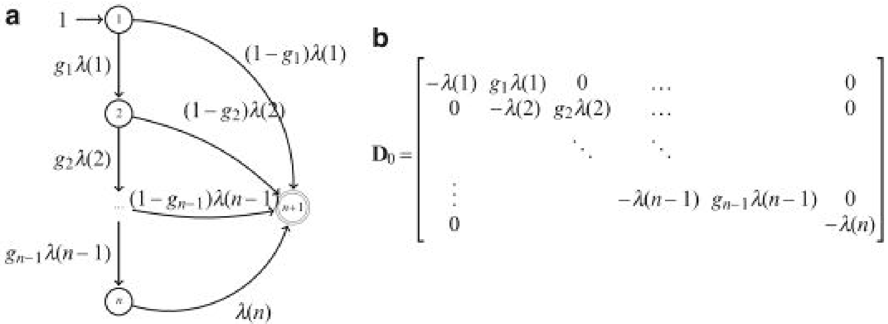


b

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda(1) & 0 & \dots & 0 & 0 \\ 0 & -\lambda(2) & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \dots & -\lambda(n-1) & 0 \\ 0 & 0 & \dots & 0 & -\lambda(n) \end{bmatrix}$$

Acyclic Phase-Type Distributions

- **Coxian Distribution**
    - **Can be considered as a mixture of hypo- and hyper-exponential distributions.**
    - **Generalized Erlang distributions with preemptive exit options.**
    - **Initial distribution vector:** $\pi = [1, 0, \ldots, 0]$



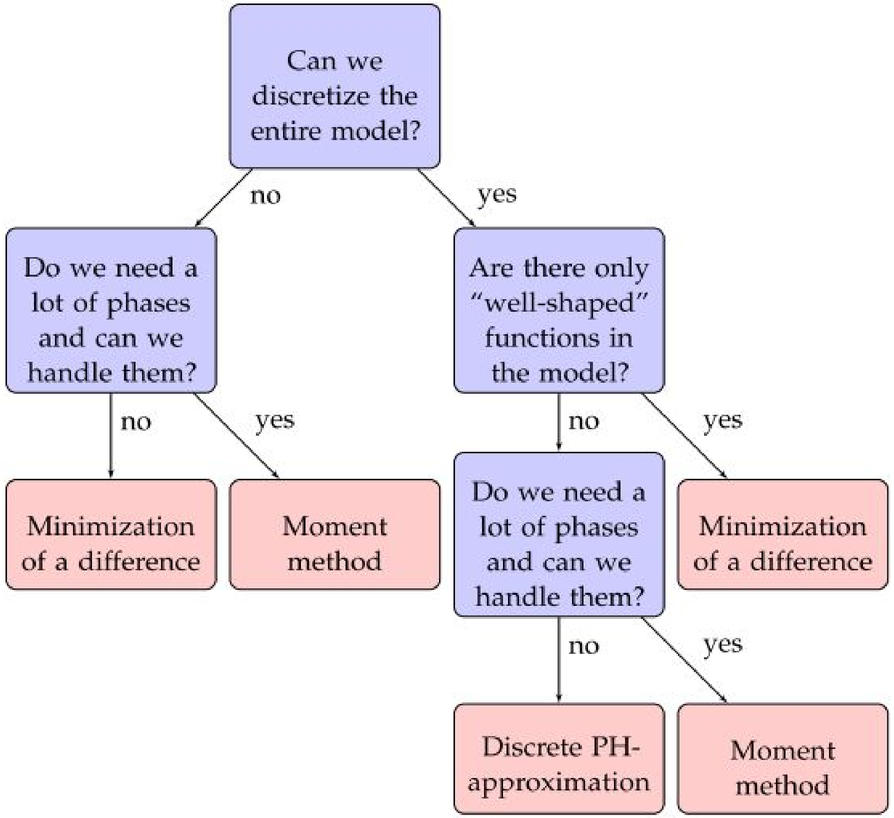
Moment Matching Methods

- **What is it?**
  - Approach for the PH-approximation
  - Member of fitting techniques:
    - they utilize incomplete information of the original distribution

Moment Matching Methods

- **Less than 3**
  - Matching only the first moment m1 is simply achieved by the exponential distribution with rate $\lambda = \frac{1}{m1}$
  - In techniques matching two moments, the coefficient of variation Cv is usually used instead of the second moment, because of consequently easier representation of results.
  - Distributions with $0 < Cv < 1$ can be aproximate by a mix of two Erlang distribuitions com γ e γ-1 phases an same rate λ.

Moment Matching Methods



Moment Matching Methods

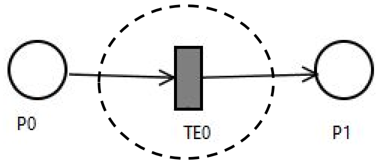
- Muitas vezes, uma atividade observada só pode ser modelada por uma variável aleatória de natureza não exponencial.
- É possível, entretanto, utilização combinações de transições exponenciais, PH-distribuitions.
- Para encontrar a distribuição, duas atividades são necessárias:
    - Determinar o tipo de aproximação necessária.
    - Encontrar os parâmetros numéricos da aproximação.
- Moments Matching algorithms fazem mapeamento de distribuições empíricas em combinações de exponenciais.

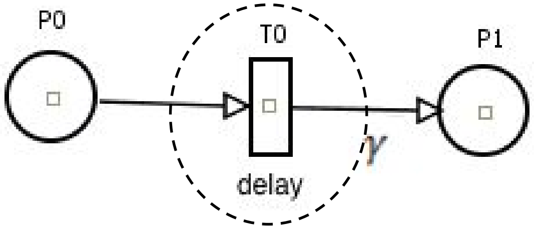
Moment Matching Methods

- Em SPN, uma abordagem para representar uma atividade com distribuição empírica é aproximá-la por uma distribuição exponencial, utilizando do primeiro momento.
    - média das durações mensuradas.
- Resultados melhores podem ser obtidos utilizando outros algoritmos que matching mais momentos.

Moment Matching Methods

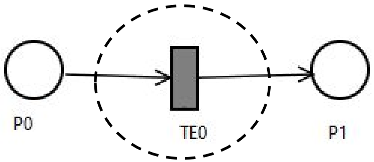
- Dado a média (μD) e o desvio padrão (σD) de uma distribuição empírica:
  - Se Cv = 1, então uma transição **exponencial** é suficiente, com um único parâmetro λ.
  - $\lambda = \dfrac{1}{\mu D}$

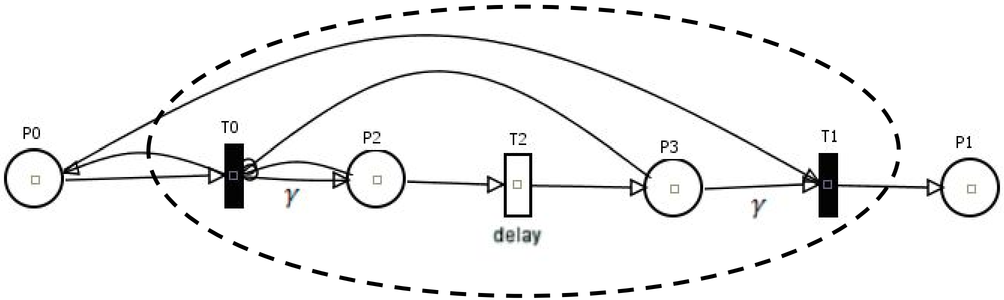




Moment Matching Methods

- Dado a média (μD) e o desvio padrão (σD) de uma distribuição empírica:
  - Se $Cv < 1$ e $Cv^{-1} \in \mathbf{Z}$, então aproxima-se para uma distribuição de **Erlang**, e é necessário estimar dois parâmetros, λ, γ.
  - $\gamma = Cv^{-2}$
  - $\lambda = \dfrac{\gamma}{\mu D}$



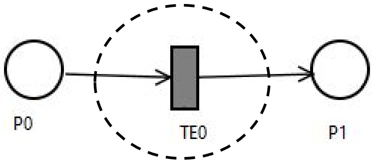


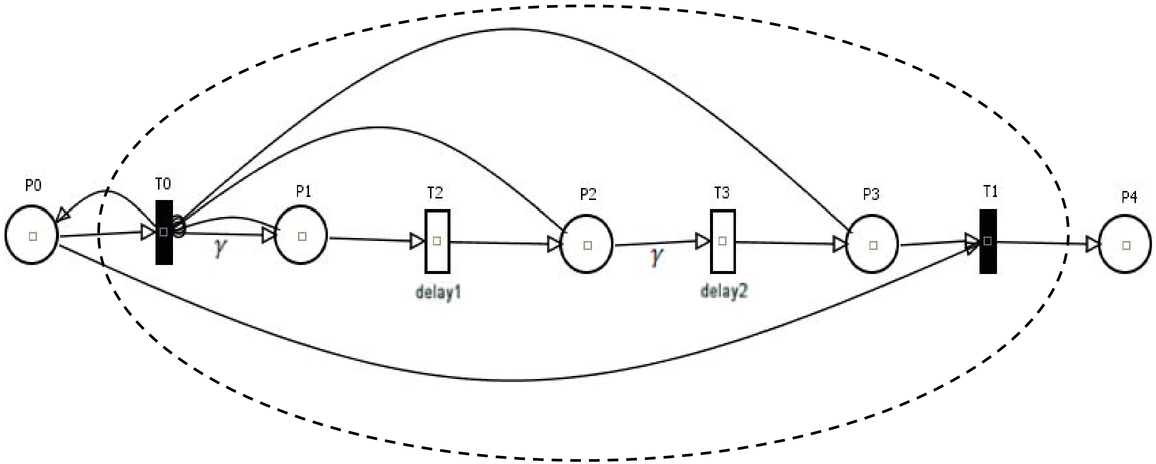
Moment Matching Methods

- Dado a média (μD) e o desvio padrão (σD) de uma distribuição empírica:
  - Se $Cv < 1$ e $Cv^{-1} \notin$ **Z**, então aproxima-se para uma distribuição de **Hipoexponencial** e é necessário estimar três parâmetros: γ, λ_1 e λ_2.
  - $Cv^{-2} - 1 \leq \gamma < Cv^{-2}$
  - $\lambda_1 = \dfrac{\gamma+1}{\mu D \pm \sqrt{\gamma(\gamma+1)\sigma D^2 - \gamma \mu D^2}}$
  - $\lambda_2 = \dfrac{\gamma+1}{\gamma \mu D \pm \sqrt{\gamma(\gamma+1)\sigma D^2 - \gamma \mu D^2}}$

Moment Matching Methods

- Hipoexponencial



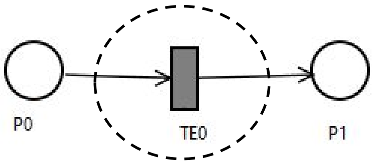


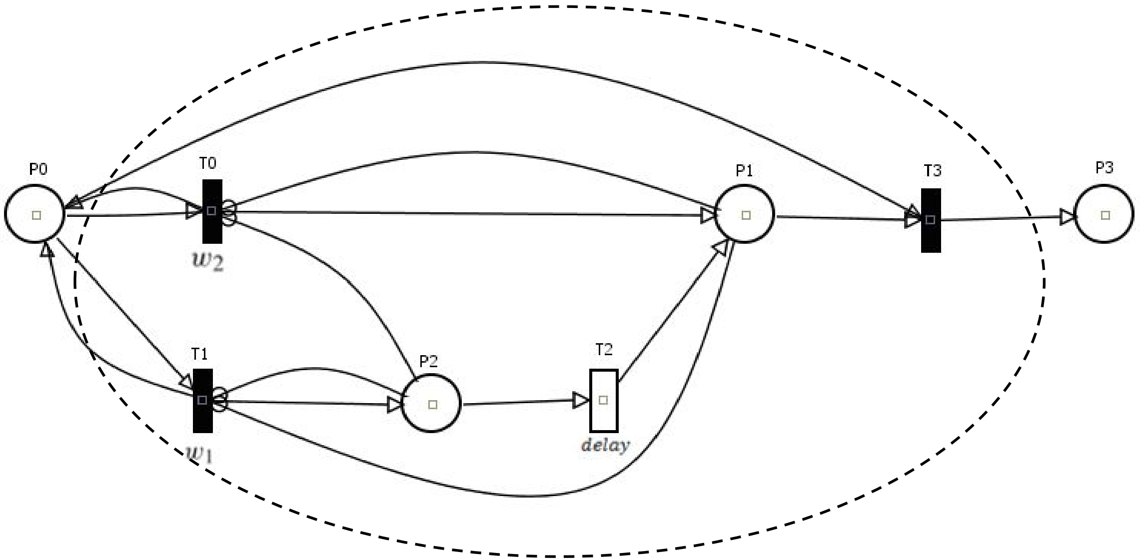
Moment Matching Methods

- Dado a média (μD) e o desvio padrão (σD) de uma distribuição empírica:
  - Se Cv > 1, então aproxima-se para uma distribuição de **Hiper-exponencial** e é necessário estimar três parâmetros: W_1, W^2 e λ_h.
  - $W_1 = \frac{2\mu D^2}{\mu D^2 + \sigma D^2}$
  - $W_2 = 1 - W_1$
  - $\lambda_h = \frac{2\mu D}{\mu D^2 + \sigma D^2}$

Moment Matching Methods

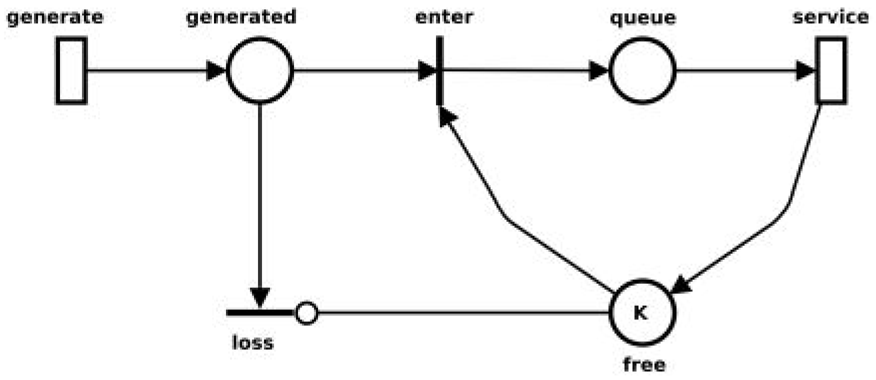
- Hiperexponencial

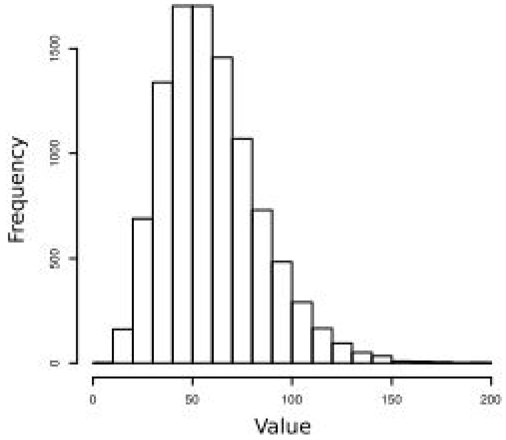




Stochastic Petri Net

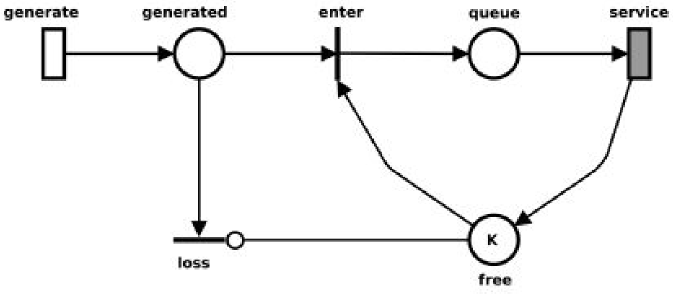
- **Service time not exponential**





Stochastic Petri Net

- **Using phase-type...**
- **MSL (Mercury)**



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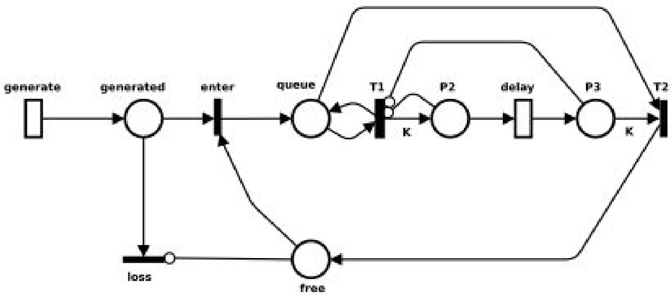

(b)

Articles

The Application of Phase Type Distributions for Modelling Queuing Systems

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ABSTRACT

Queuing models are important tools for studying the performance of complex systems, but despite the substantial queuing theory literature, it is often necessary to use approximations in the case the system is non-markovian. Phase type distribution is by now indispensable tool in creation of queuing system models. The purpose of this paper is to suggest a method and software for evaluating queuing approximations. A numerical queuing model with priorities is used to explore the behaviour of exponential phase-type approximation of service-time distribution. The performance of queuing systems described in the event language is used for generating the set of states and transition matrix between them. Two examples of numerical models are presented – a queuing system model with priorities and a queuing system model with quality control.

Articles

The Application of Phase Type Distributions for Modelling Queuing Systems

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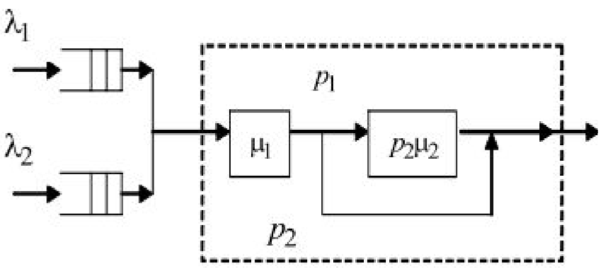

Fig. 1. *Queuing system with simple priority*

Service Time (LogNormal)

$$\begin{cases} \dfrac{1!\,p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^2}-\dfrac{\mu_2 p_1}{\mu_2^2 p_2^2}\right)=m_1;\\ \dfrac{2!\,p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^3}-\dfrac{\mu_2 p_1}{\mu_2^3 p_2^3}\right)=m_2;\\ \dfrac{3!\,p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^4}-\dfrac{\mu_2 p_1}{\mu_2^4 p_2^4}\right)=m_3;\\ p_1+p_2=1. \end{cases}$$

$$\mu_2=\frac{g_2-g_1^2}{g_1^3-2g_1g_2+g_3},\ g_k=\frac{m_k}{k!},\ k=\overline{1,3};$$

$$\mu_1=\frac{1+\mu_2 g_1\pm\sqrt{(1-\mu_2 g_1)^2+4\mu_2^{\,2}(g_2-g_1^2)}}{2g_1-2\mu_2(g_2-g_1^2)};$$

$$p_1=\frac{\mu_2(\mu_1 g_1-1)}{\mu_2(\mu_1 g_1-1)+\mu_1};$$

$$p_2=\frac{\mu_1}{\mu_2(\mu_1 g_1-1)+\mu_1}.$$

Articles

The Application of Phase Type Distributions for Modelling Queuing Systems

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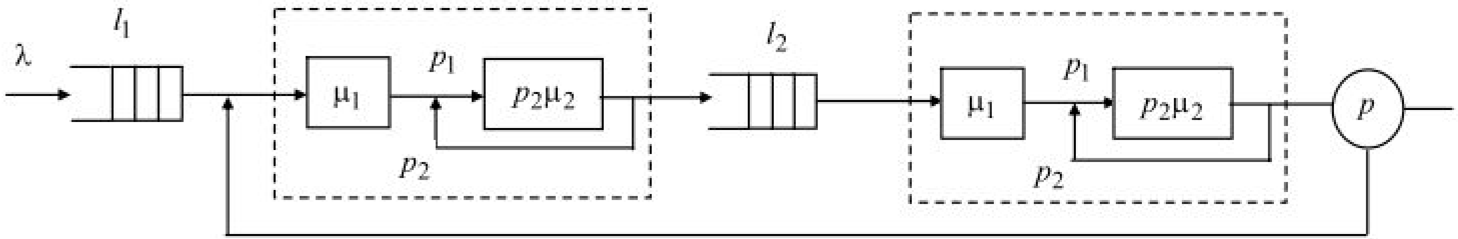

Fig.2. *Queuing system with quality control*

Articles

A Generic Method for Modeling Accelerated Life Testing Data

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Key Words: accelerated life testing, Erlang-Coxian distribution, maximum likelihood estimation

SUMMARY & CONCLUSIONS

Accelerated life testing (ALT) is widely used to expedite failures of a product in a short time period for predicting the product's reliability under normal operating conditions. The resulting ALT data are often characterized by a probability distribution, such as Weibull, Lognormal, Gamma distribution, along with a life-stress relationship. However, if the selected failure time distribution is not adequate in describing the ALT data, the resulting reliability prediction would be misleading. This paper proposes a generic method that assists engineers in modeling ALT data. The method uses Erlang-Coxian (EC) distributions, which belong to a particular subset of phase-type (PH) distributions, to approximate the underlying failure time distributions arbitrarily closely. To

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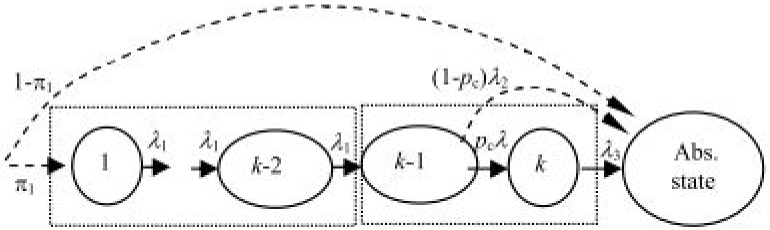
Proposed



Erlang: $E(k\text{-}2, \lambda_1)$ Coxian: $C(\lambda_2, \lambda_3, p_c)$

Figure 2 - CTMC for a k-phase EC Distribution

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Stress level	Failure times ("+": the unit is censored)				
5 volts	20.5	22.3	23.2	24.7	26
	34.1	39.6	41.8	43.6	44.9
	47.7	61.6	62.1	65.5	70.8
	87.8	118.3	120.1	145.4	157.4
	180.9	187.7	204	206.7	213.9
	215.2	218.7	254.1	262.6	293
	304	313.7	314.1	317.9	337.7
	430.2				
3.5 volts	37.8	43.6	51.1	58.6	65.5
	65.9	75.6	82.5	88.1	89
	106.6	113.1	121.1	121.5	128.3
	151.8	171.7	181	202.7	211.7
	230.7	249.9	275.6	285	296.2
	358.5	379.8	434.5	493.1	506.4
	561.1	570	577.7	876.3	922
	890+	890+	890+	941+	941+
2 volts	223.1	254	316.7	560.2	679
	737	894.4	930.5+	930.5+	
	930.5+	930.5+	930.5+	930.5+	
	930.5+	930.5+	930.5+	930.5+	
	930.5+	930.5+	930.5+	930.5+	
	930.5+	930.5+	930.5+		

Table 1 - ALT Data of Miniature Lamps

Weibull?
LogNormal?
.
.
.
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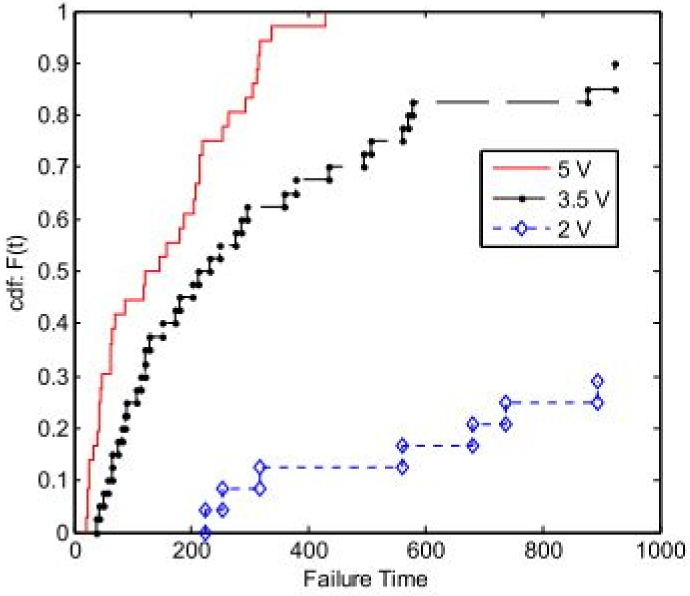

Figure 3 - Empirical cdf's of the Failure Times under Different Voltage Levels

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Values of k	MLEs of parameters	Log-likelihood lnL
3: $E(1, \lambda_1)$ & $C(\lambda_2, \lambda_3, p_c)$	$\alpha_0 = 2.9091$; $\alpha_1 = 0.5762$; $\lambda_1 = 0.0026$; $\lambda_2 = 0.0026$; $\lambda_3 = 0.0003$; $p_c = 0.6730$;	-518.5038
4: $E(2, \lambda_1)$ & $C(\lambda_2, \lambda_3, p_c)$	$\alpha_0 = 2.8807$; $\alpha_1 = 0.5730$; $\lambda_1 = 0.0045$; $\lambda_2 = 0.0045$; $\lambda_3 = 0.0003$; $p_c = 0.6980$;	-516.4058
5: $E(3, \lambda_1)$ & $C(\lambda_2, \lambda_3, p_c)$	$\alpha_0 = 2.8182$; $\alpha_1 = 0.5693$; $\lambda_1 = 0.0068$; $\lambda_2 = 0.0068$; $\lambda_3 = 0.0004$; $p_c = 0.7269$;	-515.4942
6: $E(4, \lambda_1)$ & $C(\lambda_2, \lambda_3, p_c)$	$\alpha_0 = 2.7463$; $\alpha_1 = 0.5747$; $\lambda_1 = 0.0097$;	-515.0856

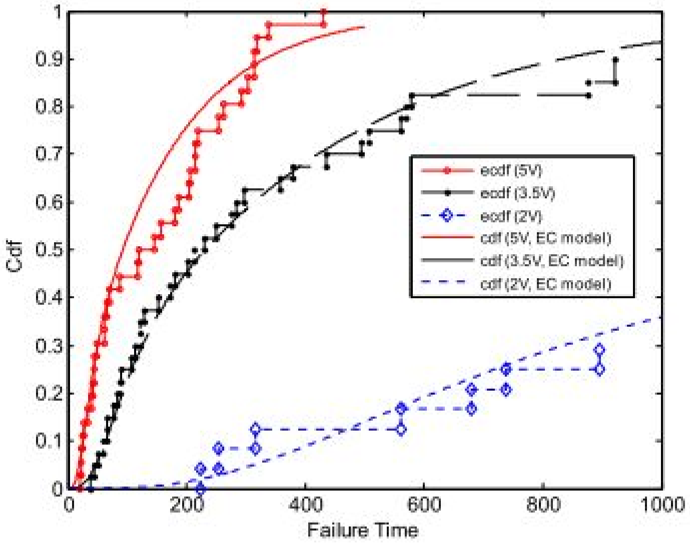

Figure 4 - Statistical Fittings of the Resulting Model

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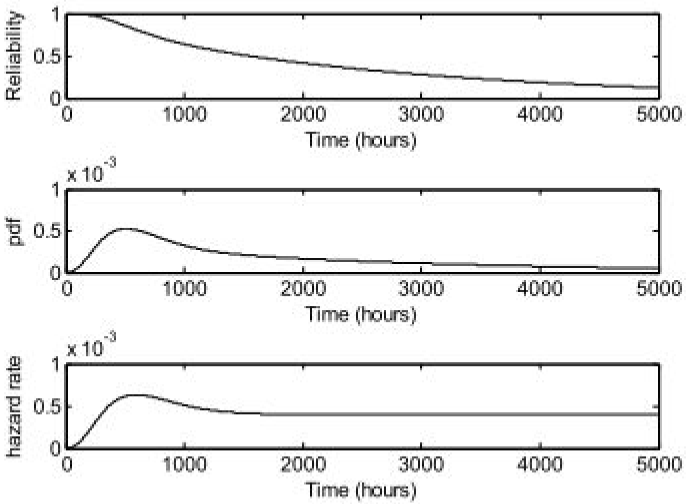

Figure 5 - Reliability Prediction using the Resulting Model (reliability function, pdf, hazard rate under 2V)

Practical Lesson

Definition - PH Distribution

- **R** - Package actuar
  - Density Phase Type
    - `dphtype(x, prob, rates)`
  - Distribution Function
    - `pphtype(q, prob, rates)`
  - Random Generator
    - `rphtype(n, prob, rates)`
  - Raw moments
    - `mphtype(order, prob, rates)`
  - Moment Generator Function
    - `mgfphtype(x, prob, rates)`

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